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INCIDENCES OF STRAIGHT LINES AND PLANE ALGEBRAIC CURVES AND SURFACES GENERATED BY THEM.

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1. INTRODUCTION.

According to Lüroth* the analytic method of investigating problems of this sort, as indicated by Clebsch,† on account of its complexity, is not practicable. Thus, even for the simple problems connected with incidences of straight lines and conics in ordinary space, Lüroth prefers to use a purely geometric method for their solution.

Some of these problems, involving merely the number of solutions in each case, have subsequently also been solved by the methods of enumerative geometry. But, without wishing to detract from the great value of this geometry, it must be said that the applications of its processes are purely mechanical and do not afford a clear insight into the geometric organism. For example, enumerative geometry does not teach how to get the solutions effectively, or constructively; it is satisfied with getting their number.

In this paper I shall show that the particular problems considered by Lüroth, and more general ones, can, after all, be solved by a relatively very simple analytical method.

2. INCIDENCE OF POINT, STRAIGHT LINE, PLANE, AND SURFACE.

Let $(a) \equiv (a_1, a_2, a_3, a_4)$ and $(b) \equiv (b_1, b_2, b_3, b_4)$, be two points determining a straight line l (hereafter we shall use simply the term *line*, meaning straight line). The homogeneous coördinates of this line are

$$\rho p_{ik} = a_i b_k - a_k b_i, \quad i, k = 1, 2, 3, 4; i \neq k,$$

so that the p_{ik} 's satisfy the identity

$$p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23} \equiv 0.$$

The line l intersects a plane (α)

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4 = 0$$

* "Ueber die Anzahl der Kegelschnitte, welche acht Geraden im Raume schneiden," *Crelle's Journal*, Vol. 68, pp. 185-192 (1868).

† *Ibid.*, Vol. 59, p. 1 ff.

in a point $(x) \equiv (x_1, x_2, x_3, x_4)$, whose coordinates are easily found as

$$(1) \quad \begin{aligned} \rho x_1 &= 0 + \alpha_2 p_{12} + \alpha_3 p_{13} + \alpha_4 p_{14}, \\ \rho x_2 &= -\alpha_1 p_{12} + 0 + \alpha_3 p_{23} + \alpha_4 p_{24}, \\ \rho x_3 &= -\alpha_1 p_{13} - \alpha_2 p_{23} + 0 + \alpha_4 p_{34}, \\ \rho x_4 &= -\alpha_1 p_{14} - \alpha_2 p_{24} - \alpha_3 p_{34} + 0. \end{aligned}$$

Incidentally, it is noticed that this represents a singular null-system, whose determinant is the square of the lefthand side of the identity satisfied by the coördinates of the line, and in which to every plane (α) not passing through l corresponds its point of intersection with l .

If we now substitute (1) in the point equation of a surface S_n of order n

$$(2) \quad f(x_1, x_2, x_3, x_4) = 0,$$

we obtain an equation

$$(3) \quad F(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = 0,$$

which is homogeneous and of order n in the α 's, and also in the p_{ik} 's, and of order 1 in the coefficients of (2). The values of (1) which satisfy (2) are evidently the coördinates of a point on the surface S_n , but, being in the form (1), they are evidently also the coördinates of a point on l and (α) , and, consequently, the coordinates of a point of intersection of l and S_n . Any plane (α) which passes through a point of intersection of l and S_n , will therefore satisfy (3). Conversely, every solution (β) of (3) represents a plane through one of the points of intersection of l and S_n . This follows immediately from the identity

$$F(\beta_1, \beta_2, \beta_3, \beta_4) \equiv f(\beta_2 p_{12} + \beta_3 p_{13} + \beta_4 p_{14}, \dots),$$

in which the point $\rho x_1 = 0 + \beta_2 p_{12} + \beta_3 p_{13} + \beta_4 p_{14}, \dots$ evidently lies on S_n , (β) and l . From this follows that the lefthand side of (3) resolves into n linear factors in the α 's, which, set equal to zero, represent the equations of the intersections of l and S_n , or the bundles of planes through these points. (3) is therefore a necessary and sufficient condition for the incidence of point, line, plane, and surface.

When a plane (α) passes through l then its coordinates expressed by the coordinates of l , and a parameter λ are

$$(4) \quad \begin{aligned} \rho \alpha_1 &= -p_{34}, \\ \rho \alpha_2 &= \lambda p_{34}, \\ \rho \alpha_3 &= \lambda p_{42} + p_{14}, \\ \rho \alpha_4 &= \lambda p_{23} - p_{13}. \end{aligned}$$

For these values of the α 's, the coördinates of (x) given by (1) vanish identically, i.e., the point (x) as the intersection of (α) and l is indetermined.

As (2) is of degree n , and as each x in (2) vanishes when the α 's have the values (4), it is evident that for every set of values (4), determined by λ , (2) vanishes to the n th order; in other words, *every plane, determined by λ in (4), is an n -fold plane of the degenerate surface (3) of class n .* This is also geometrically evident. (3) consists of the n bundles of planes through the n points of intersection of l with S_n , and a plane through l is common to all n bundles, and is therefore n -fold in the totality of bundles as given by (3).

3. APPLICATION TO PLANE n -ICS IN SPACE CUTTING A FIXED INDEPENDENT PLANE n -IC IN n POINTS. THE SURFACE E .

Considering general curves and surfaces, if not stated otherwise, a surface S_n of order n which passes through a fixed plane n -ic C_n^0 depends on

$$\frac{(n+1)(n+2)(n+3)}{6} - 1 - \frac{n(n+3)}{2}$$

parameters. Let C_n be another plane n -ic cutting each of $[n(n+3)/2] - n$ lines l , and the C_n^0 in n collinear points. If S_n is also made to contain C_n , the number of independent parameters left for the complete determination of S_n is

$$\frac{(n+1)(n+2)(n+3)}{6} - 1 - \frac{n(n+3)}{2} - \frac{n(n+3)}{2} + n = \frac{(n-1)n(n+1)}{6}.$$

Choosing this as the number of fixed points P through which S_n shall pass, imposes upon S_n (containing C_n^0)

$$\frac{n(n+3)}{2} + \frac{(n-1)n(n+1)}{6} = \frac{n(n^2+3n+8)}{6}$$

given conditions, so that there are

$$\frac{(n+1)(n+2)(n+3)}{6} - 1 - \frac{n(n^2+3n+8)}{6} = \frac{n(n+1)}{2}$$

independent parameters left. Hence there are $[n(n+1)/2] + 1$ linearly independent surfaces $\phi_1, \phi_2, \phi_3, \dots, \phi_{[n(n+1)/2]+1}$ through C_n^0 and the $n(n^2-1)/6$ fixed points P , and the general S_n may be written in the form

$$(5) \quad a_1\phi_1 + a_2\phi_2 + a_3\phi_3 + \dots + a_{[n(n+1)/2]+1} \cdot \phi_{[n(n+1)/2]+1} = 0.$$

Every plane (α) cuts the C_n^0 and the $n(n+1)/2$ lines l in points of a plane n -ic C_n which together with C_n^0 and the fixed points P determine an S_n uniquely. But when (α) passes through one of the points P , the corresponding S_n degenerates into the plane (α) itself, and an $(n-1)$ -ic.

Now let l_m be one of $[n(n+1)/2] + 1$ independent lines l , and denote its

homogeneous coördinates by p_{ik}^m . Substituting these values in (1) gives the coördinates of the point of intersection (x) of the line l_m with the plane (α). Let $\phi_1^m, \phi_2^m, \phi_3^m, \dots$ be the result of the substitution of the coördinates of (x) in $\phi_1, \phi_2, \phi_3, \dots$; then

$$(6) \quad a_1\phi_1^m + a_2\phi_2^m + a_3\phi_3^m + \dots + a_{[n(n+1)/2]+1}\phi_{[n(n+1)/2]+1}^m = 0$$

is the condition that the point of intersection of l_m and (α) lies on S_n . Repeating this process for all lines l_m , $m = 1, 2, 3, \dots, [n(n+1)/2] + 1$; and the same (α), leads to $[n(n+1)/2] + 1$ equations of order n in the α 's, whose consistency is assured when the determinantal equation

$$(7) \quad \begin{vmatrix} \phi_1^1 & \phi_2^1 & \phi_3^1 & \dots & \phi_{[n(n+1)/2]+1}^1 \\ \phi_1^2 & \phi_2^2 & \phi_3^2 & \dots & \phi_{[n(n+1)/2]+1}^2 \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \phi_1^{n(n+1)/2+1} & \phi_2^{n(n+1)/2+1} & \phi_3^{n(n+1)/2+1} & \dots & \phi_{[n(n+1)/2]+1}^{n(n+1)/2+1} \end{vmatrix} = 0$$

is satisfied. This is an equation of order

$$n \left(\frac{n(n+1)}{2} + 1 \right) = \frac{n^3 + n^2 + 2n}{2}$$

in the α 's. But the bundles of planes through the fixed points P , determining degenerate S_n 's must be subtracted, so that the degree of the equation proper is

$$\frac{n^3 + n^2 + 2n}{2} - \frac{n(n^2 - 1)}{6} = \frac{n(2n^2 + 3n + 7)}{6}.$$

Every plane (α) of the reduced equation (7) cuts the C_n^0 in n points and the $[n(n+1)/2] + 1$ lines l in points which lie on a plane n -ic C_n . Hence

THEOREM 1. *The planes, each of which cuts singly each of $[n(n+1)/2] + 1$ lines l and an independent plane n -ic C_n^0 together in a system of points which lies on a plane n -ic C_n , form a surface E of class*

$$\frac{n(2n^2 + 3n + 7)}{6}.$$

4. DEVELOPABLE SURFACE D WHOSE PLANES CUT A FIXED PLANE n -IC AND $[n(n+1)/2] + 2$ INDEPENDENT LINES IN POINTS OF PLANE n -ICS.

Let E_1 be the surface determined by the lines

$$l_1, l_2, l_3, \dots, l_{n(n+1)/2}, l_{[n(n+1)/2]+1};$$

E_2 the surface determined by the lines

$$l_1, l_2, l_3, \dots, l_{n(n+1)/2}, l_{[n(n+1)/2]+2},$$

according to theorem (1), E_1 and E_2 have a developable surface D^* of class

$$\left[\frac{n(2n^2 + 3n + 7)}{6} \right]^2$$

in common. But through each of the $n(n+1)/2$ lines common to both E_1 and E_2 there is a pencil of planes, whose planes cut the remaining lines and C_n^0 in points which lie on an n -ic for each such plane. These planes are each n -fold in E_1 and E_2 , and as they all contain n -ics which cut one of the lines l improperly, each pencil of planes through the lines common to E_1 and E_2 must be deducted n^2 -fold from D^* . This leaves for the class of the developable surface D proper the number

$$\left[\frac{n(2n^2 + 3n + 7)}{6} \right]^2 - n^2 \frac{n(n+1)}{2} = \frac{n^2(4n^4 + 12n^3 + 19n^2 + 24n + 49)}{36}.$$

Hence

THEOREM 2. *The planes, each of which cuts singly each of $[n(n+1)/2] + 2$ lines l and an independent plane n -ic C_n^0 together in a system of points which lies on a plane n -ic C_n , form a surface D of class*

$$\frac{n^2(4n^4 + 12n^3 + 19n^2 + 24n + 49)}{36}.$$

5. NUMBER OF PLANE n -ICS CUTTING SINGLY EACH OF $[n(n+1)/2] + 3$ LINES l AND A FIXED PLANE n -IC IN n POINTS.

The surface D determined by the lines

$$l_1, l_2, l_3, \dots, l_{n(n+1)/2}, l_{[n(n+1)/2]+1}, l_{[n(n+1)/2]+2},$$

according to theorem (2), and the surface E determined by the lines

$$l_1, l_2, l_3, \dots, l_{n(n+1)/2}, l_{[n(n+1)/2]+3},$$

according to theorem (1), have

$$\frac{n^2(4n^4 + 12n^3 + 19n^2 + 24n + 49)}{36} \cdot \frac{n(2n^2 + 3n + 7)}{6}$$

common solutions, including the improper ones. To get the number of the latter we must take into account that there are $n(2n^2 + 3n + 7)/6$ planes through each of the $n(n+1)/2$ lines common to D and E , of which each cuts the remaining $[n(n+1)/2] + 2$ lines of a plane n -ic belonging to both D and E . Each of these planes is a multiple plane of order n in both surfaces. The number of proper planes cutting singly all $[n(n+1)/2] + 3$ lines in points on plane n -ics is therefore

$$\frac{n^2(4n^4 + 12n^3 + 19n^2 + 24n + 49)}{36} \cdot \frac{n(2n^2 + 3n + 7)}{6}$$

$$\begin{aligned}
& - \frac{n(n+1)}{2} \cdot \frac{n(2n^2+3n+7)}{6} \cdot n^2 \\
& = \frac{n^3(8n^6+36n^5+66n^4+99n^3+123n^2+89n+343)}{216}.
\end{aligned}$$

Denoting this number by N , the result may be stated in

THEOREM 3. *There are N plane n -ics which cut singly each of $[n(n+1)/2] + 3$ lines and which cut, in each case, a fixed plane n -ic in n points.*

6. A CERTAIN SURFACE GENERATED BY PLANE n -ICS.

Given a fixed plane n -ic C_n^0 and $n(n+1)/2$ independent lines l . Consider the pencil of planes through another independent fixed line s . Every plane of this pencil cuts C_n^0 and the l 's in points on an n -ic C_n . As the plane describes the pencil through s , the corresponding plane n -ic C_n describes a certain surface, whose order is determined by the number of points in which any other independent line, say $l_{[n(n+1)/2]+1}$ cuts the surface. Now we know the class of the surface E determined by $l_1, l_2, l_3, \dots, l_{n(n+1)/2}, l_{[n(n+1)/2]+1}$. The number of plane n -ics through s cutting all these $[n(n+1)/2] + 1$ lines l is, of course, equal to the class of E . Hence $l_{n(n+1)/2+1}$ cuts the surface in a number of points equal to the class of E , and we have

THEOREM 4. *The plane n -ics, whose planes form a pencil, and which cut a fixed plane n -ic in n points, and which cut singly each of $n(n+1)/2$ lines, generate a surface of order*

$$\frac{n(2n^2+3n+7)}{6},$$

in which the axis s of the pencil is an

$$\left[\frac{n(2n^2+3n+7)}{6} - n \right] = \frac{n(2n^2+3n+1)}{6}$$

fold line.

The last part of this theorem is obvious, since a plane through s cuts the surface, outside of s , in a plane n -ic only.

7. EXAMPLE OF CIRCLES.

Let the surfaces S_n be spheres, so that the fixed plane n -ic C_n^0 is the imaginary sphero-circle (absolute) at infinity. In this case (as well as for any other conic as C_n^0) $[n(n+1)/2] + 3 = 6$, $N = 175$, and theorem 3 assumes the form of a corollary:

There are 175 circles cutting singly each of 6 independent lines.

8. GENERALIZATION OF LÜROTH'S PROBLEM.

In the paper referred to in the introduction Lüroth investigates incidences of conics and lines and certain surfaces enveloped by the planes of such conics, or generated by such conics. His concluding theorem states that there are 92 conics which cut singly each of 8 independent lines in space. I shall generalize this result by establishing the number of plane n -ics which cut singly each of $[n(n+3)/2] + 3$ independent lines in space.

The key for the solution of this problem is again furnished by the establishment of the class of the surface E formed by the planes of the n -ics cutting singly each of $[n(n+3)/2] + 1$ lines l . For this purpose I shall follow briefly a line of reasoning entirely similar to that for the surface E in § 3. Accordingly it is found that there are $(n^2 + n + 2)/2$ linearly independent surfaces $\phi_1, \phi_2, \phi_3, \dots$ of order n which pass through n definite fixed lines l and $(n^3 - 3n^2 + 2n)/6$ independent fixed points P . Hence any surface S_n of order n through these fixed elements may be represented by a linear polynomial

$$S_n = a_1\phi_1 + a_2\phi_2 + a_3\phi_3 + \dots$$

in the ϕ 's. The condition that a plane (α) shall cut $(n^2 + n + 2)/2$ other independent lines l in points which lie on an S_n is equivalent with the same number of equations of the form

$$a_1\phi_1^m + a_2\phi_2^m + a_3\phi_3^m + \dots = 0.$$

This leads to a determinantal equation of order $n(n^2 + n + 2)/2$ in the α 's. But, again, the bundles of planes through the points P must be deducted, so that a reduced equation of degree

$$\frac{n(n^2 + n + 2)}{2} - \frac{n^3 - 3n^2 + 2n}{6} = \frac{n^3 + 3n^2 + 2n}{6}$$

in the α 's is left. Thus, in analogy with theorem (1) we have

THEOREM 5. *The planes each of which cuts singly each of $[n(n+3)/2] + 1$ independent fixed lines in space in points which lie on a plane n -ic is a surface E of class*

$$\frac{n^3 + 3n^2 + 2n}{3}.$$

By Luroth's method, extended to the case of n -ics, and in analogy with the procedure in §§ 4 and 5, it is found that the class of the developable surface D , formed by the planes which cut $[n(n+3)/2] + 2$ lines l in points on n -ics, is

$$\frac{n^2(2n^4 + 12n^3 + 17n^2 - 3n + 8)}{18}.$$

In case of conics, $n = 2$, this number, as shown by Lüroth, has to be reduced by 10, so that for conics the class of D is 34.

Finally the number of plane n -ics cutting $[n(n + 3)/2] + 3$ independent lines in space is

$$\frac{n^3(n^2 + 3n + 2)(n^4 + 6n^3 + 4n^2 - 15n + 4)}{27}.$$

For $n = 2$ this number is reduced by 20 (Lüroth, loc. cit.) so that the number of conics cutting singly each of 8 lines is 92.